On marginals of Gaussian random vectors

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Abstract

Consider a Gaussian random vector

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{\mathrm{M}} \\ \boldsymbol{X}_{\mathrm{N}} \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{\mathrm{M}} \\ \boldsymbol{\mu}_{\mathrm{N}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{MM}} & \boldsymbol{\Sigma}_{\mathrm{MN}} \\ \boldsymbol{\Sigma}_{\mathrm{NM}} & \boldsymbol{\Sigma}_{\mathrm{NN}} \end{bmatrix} \right),$$

where $\boldsymbol{X}_{\mathrm{M}}$ and $\boldsymbol{X}_{\mathrm{N}}$ denote subsets of entries of $\boldsymbol{X}$ and the mean and covariance matrix are partitioned consistent with partitioning of $\boldsymbol{X}$. It is well-known, or at least it should be, that marginals of $\boldsymbol{X}$ are Gaussian, with $\boldsymbol{X}_{\mathrm{M}} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathrm{M}}, \boldsymbol{\Sigma}_{\mathrm{MM}})$ and $\boldsymbol{X}_{\mathrm{N}} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathrm{N}}, \boldsymbol{\Sigma}_{\mathrm{NN}})$. In this note, we provide three proofs of this fact: one is done by computing the marginal density directly, and the other two are short proofs that use further properties of multivariate Gaussian distribution.

1 Introduction

Consider a d-dimensional Gaussian random vector $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The covariance matrix $\boldsymbol{\Sigma}$ is symmetric, and is assumed to positive definite throughout. Suppose we partition $\boldsymbol{X}$ according to

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{\mathrm{M}} \\ \boldsymbol{X}_{\mathrm{N}} \end{bmatrix}, \tag{1.1}$$

with $\boldsymbol{X}_{\mathrm{M}} \in \mathbb{R}^m$, where $m < d$, and $\boldsymbol{X}_{\mathrm{N}} \in \mathbb{R}^n$, $n = d - m$. With no loss of generality, we can take $\boldsymbol{X}_{\mathrm{M}}$ to be the first m elements of $\boldsymbol{X}$, and $\boldsymbol{X}_{\mathrm{N}}$ the rest. We partition the mean vector and the covariance matrix accordingly

$$\begin{bmatrix} \boldsymbol{X}_{\mathrm{M}} \\ \boldsymbol{X}_{\mathrm{N}} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{\mathrm{M}} \\ \boldsymbol{\mu}_{\mathrm{N}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{MM}} & \boldsymbol{\Sigma}_{\mathrm{MN}} \\ \boldsymbol{\Sigma}_{\mathrm{NM}} & \boldsymbol{\Sigma}_{\mathrm{NN}} \end{bmatrix} \right).$$

The probability density function (PDF) of $\boldsymbol{X}$ is

$$f(\boldsymbol{x}_{\mathrm{M}}, \boldsymbol{x}_{\mathrm{N}}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2} \begin{bmatrix} \boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}} \\ \boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{MM}} & \boldsymbol{\Sigma}_{\mathrm{MN}} \\ \boldsymbol{\Sigma}_{\mathrm{NM}} & \boldsymbol{\Sigma}_{\mathrm{NN}} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}} \\ \boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}} \end{bmatrix} \right).$$

The marginal PDF of $\boldsymbol{X}_{\mathrm{M}}$, which defines the distribution law of $\boldsymbol{X}_{\mathrm{M}}$, is

$$f_{\mathrm{M}}(\boldsymbol{x}_{\mathrm{M}}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}_{\mathrm{M}}, \boldsymbol{x}_{\mathrm{N}})\, d\boldsymbol{x}_{\mathrm{N}}.$$

Below, we prove the following result:

Theorem 1.1. $X_M \sim \mathcal{N}(\boldsymbol{\mu}_M, \boldsymbol{\Sigma}_{MM})$.

We provide a direct proof by computing the marginal PDF f_{M} in Section 2. The argument presented there is of course standard and has been presented by many authors. The proof presented in Section 2 follows in similar lines as the argument given in [2]. Then, in Section 3, we discuss alternative, more elegant proofs, that rely on further properties of multivariate Gaussian distribution.

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Last revised: November 5, 2018

2 The basic approach

We discuss some preliminaries, before stating the proof of Theorem 1.1.

Gaussian PDF. Consider an n-dimensional gaussian random variable $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{C})$. The covariance matrix $\mathbf{C}$ is symmetric, and is assumed to be positive definite, in which case the distribution law of $\boldsymbol{Z}$ admits a probability density function (PDF) given by

$$f(\boldsymbol{z}) = \frac{1}{(2\pi)^{n/2}|\mathbf{C}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\boldsymbol{z}-\boldsymbol{\mu})\right), \quad \boldsymbol{z} \in \mathbb{R}^n.$$

By definition, the PDF must integrate to one; thus, in particular

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\boldsymbol{z}-\boldsymbol{\mu})\right) d\boldsymbol{z} = (2\pi)^{n/2}|\mathbf{C}|^{1/2}. \tag{2.1}$$

Completing the Square. When manipulating multivariate Gaussians, the basic idea of completing a square comes up often. This is recorded in the following lemma:

Lemma 2.1. *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be symmetric positive definite and let $\boldsymbol{z}, \boldsymbol{b}$, and $\boldsymbol{c}$ be in $\mathbb{R}^n$. Then,*

$$\frac{1}{2}\boldsymbol{z}^T\mathbf{A}\boldsymbol{z} + \boldsymbol{b}^T\boldsymbol{z} + \boldsymbol{c} = \frac{1}{2}\left(\boldsymbol{z}+\mathbf{A}^{-1}\boldsymbol{b}\right)^T\mathbf{A}\left(\boldsymbol{z}+\mathbf{A}^{-1}\boldsymbol{b}\right) + \boldsymbol{c} - \frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b}. \tag{2.2}$$

Proof. This is seen by direct calculation.

$$\begin{aligned}
\frac{1}{2}\boldsymbol{z}^T\mathbf{A}\boldsymbol{z} + \boldsymbol{b}^T\boldsymbol{z} + \boldsymbol{c} &= \frac{1}{2}\boldsymbol{z}^T\mathbf{A}\boldsymbol{z} + \frac{1}{2}\boldsymbol{b}^T\boldsymbol{z} + \frac{1}{2}\boldsymbol{z}^T\boldsymbol{b} + \boldsymbol{c} + \left(\frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b} - \frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b}\right)\\
&= \frac{1}{2}\boldsymbol{b}^T\boldsymbol{z} + \frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b} + \frac{1}{2}\boldsymbol{z}^T\mathbf{A}\boldsymbol{z} + \frac{1}{2}\boldsymbol{z}^T\boldsymbol{b} + \boldsymbol{c} - \frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b}\\
&= \frac{1}{2}\left(\boldsymbol{b}^T\mathbf{A}^{-1}\mathbf{A} + \boldsymbol{z}^T\mathbf{A}\right)\left(\boldsymbol{z}+\mathbf{A}^{-1}\boldsymbol{b}\right) + \boldsymbol{c} - \frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b}\\
&= \frac{1}{2}\left(\boldsymbol{z}+\mathbf{A}^{-1}\boldsymbol{b}\right)^T\mathbf{A}\left(\boldsymbol{z}+\mathbf{A}^{-1}\boldsymbol{b}\right) + \boldsymbol{c} - \frac{1}{2}\boldsymbol{b}^T\mathbf{A}^{-1}\boldsymbol{b}. \quad \square
\end{aligned}$$

Proof 1 of Theorem 1.1. Consider the marginal PDF of X_{M}

$$\begin{aligned}
f(\boldsymbol{x}_{\mathrm{M}}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x}_{\mathrm{M}}, \boldsymbol{x}_{\mathrm{N}})\, d\boldsymbol{x}_{\mathrm{N}}\\
&= \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\begin{bmatrix}\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}}\\ \boldsymbol{x}_{\mathrm{N}}-\boldsymbol{\mu}_{\mathrm{N}}\end{bmatrix}^T \begin{bmatrix}\boldsymbol{\Sigma}_{\mathrm{MM}} & \boldsymbol{\Sigma}_{\mathrm{MN}}\\ \boldsymbol{\Sigma}_{\mathrm{NM}} & \boldsymbol{\Sigma}_{\mathrm{NN}}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}}\\ \boldsymbol{x}_{\mathrm{N}}-\boldsymbol{\mu}_{\mathrm{N}}\end{bmatrix}\right) d\boldsymbol{x}_{\mathrm{N}}.
\end{aligned} \tag{2.3}$$

Let

$$Q = (2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2},$$

and note that that by the formula for the determinant of a block matrix [6],

$$|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{\mathrm{MM}}||\boldsymbol{\Sigma}_{\mathrm{NN}} - \boldsymbol{\Sigma}_{\mathrm{NM}}\boldsymbol{\Sigma}_{\mathrm{MM}}^{-1}\boldsymbol{\Sigma}_{\mathrm{MN}}|. \tag{2.4}$$

For convenience, we introduce the notation

$$\mathbf{S} = \begin{bmatrix}\mathbf{S}_{\mathrm{MM}} & \mathbf{S}_{\mathrm{MN}}\\ \mathbf{S}_{\mathrm{NM}} & \mathbf{S}_{\mathrm{NN}}\end{bmatrix} = \boldsymbol{\Sigma}^{-1}.$$

The blocks in the definition of $\mathbf{S}$ can be computed using the formula for inverse of a block matrix [6], but for the time being, we will not need their explicit expression. Expanding (2.3) we obtain

$$\begin{aligned}
f(\boldsymbol{x}_{\mathrm{M}}) = \frac{1}{Q}\int_{\mathbb{R}^n} \exp\Bigg(-\Bigg[&\frac{1}{2}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MM}}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}}) + \frac{1}{2}(\boldsymbol{x}_{\mathrm{N}}-\boldsymbol{\mu}_{\mathrm{N}})^T\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}}) +\\
&\frac{1}{2}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MN}}(\boldsymbol{x}_{\mathrm{N}}-\boldsymbol{\mu}_{\mathrm{N}}) + \frac{1}{2}(\boldsymbol{x}_{\mathrm{N}}-\boldsymbol{\mu}_{\mathrm{N}})^T\mathbf{S}_{\mathrm{NN}}(\boldsymbol{x}_{\mathrm{N}}-\boldsymbol{\mu}_{\mathrm{N}})\Bigg]\Bigg)\, d\boldsymbol{x}_{\mathrm{N}}.
\end{aligned}$$

Completing the square, using the formula (2.2), with

$$\boldsymbol{z} = \boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}}, \quad \mathbf{A} = \mathbf{S}_{\mathrm{NN}}, \quad \boldsymbol{b} = \mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}}), \quad \boldsymbol{c} = \frac{1}{2}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MM}}(\boldsymbol{x}_{\mathrm{M}}-\boldsymbol{\mu}_{\mathrm{M}}),$$

we obtain

$$f(\boldsymbol{x}_{\mathrm{M}}) = \frac{1}{Q}\int_{\mathbb{R}^n} \exp\Bigg(- \bigg[\frac{1}{2}((\boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}}) + \mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}}))^T\mathbf{S}_{\mathrm{NN}}((\boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}}) + \mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}}))$$
$$+\frac{1}{2}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}}) - \frac{1}{2}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})\bigg]\Bigg)\, d\boldsymbol{x}_{\mathrm{N}}.$$

Factoring out the terms that do not contain $\boldsymbol{x}_{\mathrm{N}}$ we obtain

$$f(\boldsymbol{x}_{\mathrm{M}}) = \frac{1}{Q}\exp\left(\frac{1}{2}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}}) - \frac{1}{2}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})^T\mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})\right)$$
$$\int_{\mathbb{R}^n} \exp\Bigg(- \bigg[\frac{1}{2}((\boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}}) + \mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}}))^T\mathbf{S}_{\mathrm{NN}}((\boldsymbol{x}_{\mathrm{N}} - \boldsymbol{\mu}_{\mathrm{N}}) + \mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}}))\bigg]\Bigg)\, d\boldsymbol{x}_{\mathrm{N}}.$$

For clarity, we introduce that notation, $\boldsymbol{m} = \boldsymbol{\mu}_{\mathrm{N}} - \mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})$, and note that the integral in the above expression can be written as

$$\int_{\mathbb{R}^n} \exp\left(- \frac{1}{2}(\boldsymbol{y} - \boldsymbol{m})^T\mathbf{S}_{\mathrm{NN}}(\boldsymbol{y} - \boldsymbol{m})\right) d\boldsymbol{y} = (2\pi)^{n/2}|\mathbf{S}_{\mathrm{NN}}^{-1}|^{1/2},$$

where the final equality follows from (2.1). Now we are left with

$$f(\boldsymbol{x}_{\mathrm{M}}) = \frac{1}{Q}(2\pi)^{n/2}|\mathbf{S}_{\mathrm{NN}}^{-1}|^{1/2}\exp\left(\frac{1}{2}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})^T(\mathbf{S}_{\mathrm{MM}} - \mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}})(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})\right). \tag{2.5}$$

Note that the exponential term matches that of a Gaussian distribution with mean $\boldsymbol{\mu}_{\mathrm{M}}$ and covariance matrix $(\mathbf{S}_{\mathrm{MM}} - \mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}})^{-1}$. It remains to check that this covariance matrix equals $\boldsymbol{\Sigma}_{\mathrm{MM}}$ and that we have the correct normalization constant. We proceed by examining the inverse of the block matrix $\mathbf{S}$:

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\Sigma}_{\mathrm{MM}} & \boldsymbol{\Sigma}_{\mathrm{MN}}\\ \boldsymbol{\Sigma}_{\mathrm{NM}} & \boldsymbol{\Sigma}_{\mathrm{NN}}\end{bmatrix} &= \begin{bmatrix}\mathbf{S}_{\mathrm{MM}} & \mathbf{S}_{\mathrm{MN}}\\ \mathbf{S}_{\mathrm{NM}} & \mathbf{S}_{\mathrm{NN}}\end{bmatrix}^{-1}\\ &= \begin{bmatrix}(\mathbf{S}_{\mathrm{MM}} - \mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}})^{-1} & -(\mathbf{S}_{\mathrm{MM}} - \mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}})^{-1}\mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\\ -\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}}(\mathbf{S}_{\mathrm{MM}} - \mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}})^{-1} & (\mathbf{S}_{\mathrm{NN}} - \mathbf{S}_{\mathrm{NM}}\mathbf{S}_{\mathrm{MM}}^{-1}\mathbf{S}_{\mathrm{MN}})^{-1}\end{bmatrix}.\end{aligned}$$

We immediately see that

$$\boldsymbol{\Sigma}_{\mathrm{MM}} = (\mathbf{S}_{\mathrm{MM}} - \mathbf{S}_{\mathrm{MN}}\mathbf{S}_{\mathrm{NN}}^{-1}\mathbf{S}_{\mathrm{NM}})^{-1}. \tag{2.6}$$

Moreover, the normalization constant in front of (2.5) simplifies to

$$\begin{aligned}\frac{1}{Q}(2\pi)^{n/2}|\mathbf{S}_{\mathrm{NN}}^{-1}|^{1/2} &= (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2}(2\pi)^{n/2}|\mathbf{S}_{\mathrm{NN}}^{-1}|^{1/2}\\ &= (2\pi)^{-m/2}|\boldsymbol{\Sigma}_{\mathrm{MM}}|^{-1/2}|\boldsymbol{\Sigma}_{\mathrm{NN}} - \boldsymbol{\Sigma}_{\mathrm{NM}}\boldsymbol{\Sigma}_{\mathrm{MM}}^{-1}\boldsymbol{\Sigma}_{\mathrm{MN}}|^{-1/2}|\boldsymbol{\Sigma}_{\mathrm{NN}} - \boldsymbol{\Sigma}_{\mathrm{NM}}\boldsymbol{\Sigma}_{\mathrm{MM}}^{-1}\boldsymbol{\Sigma}_{\mathrm{MN}}|^{1/2}\\ &= (2\pi)^{-m/2}|\boldsymbol{\Sigma}_{\mathrm{MM}}|^{-1/2}.\end{aligned} \tag{2.7}$$

In the penultimate step we used (2.4) and also the inversion formula for the block form of $\boldsymbol{\Sigma}$ in (2.3). Combining (2.5), (2.6), (2.7), concludes the proof:

$$f(\boldsymbol{x}_{\mathrm{M}}) = (2\pi)^{-m/2}|\boldsymbol{\Sigma}_{\mathrm{MM}}|^{-1/2}\exp\left(\frac{1}{2}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})^T\boldsymbol{\Sigma}_{\mathrm{MM}}^{-1}(\boldsymbol{x}_{\mathrm{M}} - \boldsymbol{\mu}_{\mathrm{M}})\right).$$

□

3 Alternative arguments

The argument presented above regarding the marginals of a Gaussian is basic in that it uses only the definition of the marginal and the definition of Gaussian PDFs. As shown below, we can also derive the distribution law of $\boldsymbol{X}_{\mathrm{M}}$ using further properties of multivariate Gaussian distribution.

Using affine transformation of a Gaussian random vector. Let's recall the following result: let $\boldsymbol{X}$ be a d-dimensional Gaussian random vector with law $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and let $\mathbf{A} \in \mathbb{R}^{k\times d}$ and $\boldsymbol{c} \in \mathbb{R}^k$. Then, $\boldsymbol{Y} = \mathbf{A}\boldsymbol{X} + \boldsymbol{c}$ is also a Gaussian and

$$\boldsymbol{Y} \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{c}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T). \tag{3.1}$$

See e.g., [4, p. 121] for a proof. This formula can be used to give a very short proof of Theorem 1.1.

Proof 2 of Theorem 1.1. Consider the Gaussian random vector $\boldsymbol{X}$ as partitioned in (1.1), and note that $\boldsymbol{X}_\mathrm{M} = \mathbf{A}\boldsymbol{X}$, with $\mathbf{A} = \begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{0}_{m\times(d-m)} \end{bmatrix}$. Therefore, $\boldsymbol{X}_\mathrm{M} \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T) = \mathcal{N}(\boldsymbol{\mu}_\mathrm{M}, \boldsymbol{\Sigma}_\mathrm{MM})$. $\square$

This is a typical way of proving the result regarding the marginals of a Gaussian discussed herein; see also [3, p. 178], where a similar proof is presented.

Using characteristic functions. Yet another quick proof of the result on the marginals of a Gaussian can be done using characteristic functions; this is the approach used for instance in [7]. Recall that for a d-dimensional random vector $\boldsymbol{X}$, its characteristic function is given by

$$\varphi_{\boldsymbol{X}}(\boldsymbol{\xi}) = \mathbb{E}\left(\exp\left(i\boldsymbol{\xi}^T\boldsymbol{X}\right)\right), \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Here $\mathbb{E}$ denotes expectation and i is the imaginary unit. The characteristic function of a random variable uniquely characterizes its distribution law [1, 7].

It is straightforward to note that, for any d-dimensional random vector $\boldsymbol{X}$, partitioned according to $\begin{bmatrix} \boldsymbol{X}_\mathrm{M}^T & \boldsymbol{X}_\mathrm{N}^T \end{bmatrix}^T$, with $\boldsymbol{X}_\mathrm{M} = (X_1, \ldots, X_m)^T$ and $\boldsymbol{X}_\mathrm{N} = (X_{m+1}, \ldots, X_d)^T$,

$$\varphi_{\boldsymbol{X}_\mathrm{M}}(\boldsymbol{\xi}_\mathrm{M}) = \varphi_{\boldsymbol{X}}\left(\begin{bmatrix} \boldsymbol{\xi}_\mathrm{M} \\ \mathbf{0} \end{bmatrix}\right), \quad \boldsymbol{\xi}_\mathrm{M} \in \mathbb{R}^m.$$

Let $\boldsymbol{X}$ be a d-dimensional random vector; it is well-known (see e.g., [5, 7]) that $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if and only if,

$$\varphi_{\boldsymbol{X}}(\boldsymbol{\xi}) = \exp\left(i\boldsymbol{\xi}^T\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\Sigma}\boldsymbol{\xi}\right), \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Proof 3 of Theorem 1.1. For a d-dimensional Gaussian random vector $\boldsymbol{X}$ partitioned according to (1.1),

$$\varphi_{\boldsymbol{X}_\mathrm{M}}(\boldsymbol{\xi}_\mathrm{M}) = \varphi_{\boldsymbol{X}}\left(\begin{bmatrix} \boldsymbol{\xi}_\mathrm{M} \\ \mathbf{0} \end{bmatrix}\right) = \exp\left(i\boldsymbol{\xi}_\mathrm{M}^T\boldsymbol{\mu}_\mathrm{M} - \frac{1}{2}\boldsymbol{\xi}_\mathrm{M}^T\boldsymbol{\Sigma}_\mathrm{MM}\boldsymbol{\xi}_\mathrm{M}\right), \quad \boldsymbol{\xi}_\mathrm{M} \in \mathbb{R}^m,$$

from which it immediately follows that $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{x}_\mathrm{M}, \boldsymbol{\Sigma}_\mathrm{MM})$. $\square$

References

[1] R. N. Bhattacharya and E. C. Waymire. *A basic course in probability theory*, volume 69. Springer, 2007.

[2] C. Bishop. *Pattern recognition and machine learning*. Springer, 2006.

[3] B. Flury. *A first course in multivariate statistics*. Springer Science & Business Media, 2013.

[4] A. Gut. *An intermediate course in probability*. Springer Texts in Statistics. Springer, New York, second edition, 2009.

[5] S. Janson. *Gaussian Hilbert spaces*, volume 129. Cambridge university press, 1997.

[6] C. D. Meyer. *Matrix analysis and applied linear algebra*, volume 71. SIAM, 2000.

[7] Y. L. Tong. *The multivariate normal distribution*. Springer Science & Business Media, 2012.